Int. J. Heat Mass Transfer. Vol. 15. pp. 866–870. Pergamon Press 1972. Printed in Great Britain

CALCULATIONS OF THE EFFECT OF TANGENTIAL EDDY DIFFUSIVITY ON A NON-SYMMETRIC TURBULENT DIFFUSION IN A PLAIN TUBE

ALAN QUARMBY
University of Manchester Institute of Science and Technology, Manchester, England

(*Received* 22 *June* 1971 *and in revised form* 22 *October* 1971)

NOMENCLATURE

D, diffusivity;
r, radius;
r_0, outer radius of tube;
r^+, ru^*/ν;
r_0^+, r_0u^*/ν;
Re, Reynolds Number, $2u_m r_0/\nu$;
Sc, Schmidt Number, ν/D;
u, velocity;
u^*, $\sqrt{(\tau_0/\rho)}$;
u_m, bulk mean velocity;
x, axial distance;
x^+, $x/2r_0$.

Greek symbols
ε, eddy diffusivity;
τ_0, shear stress at wall;
θ, dimensionless concentration;
ρ, density;
ν, kinematic viscosity;
ω, angular co-ordinate.

Subscripts
h, heat;
d, mass;
r, radial;
ω, tangential.

INTRODUCTION

UNTIL quite recently, most work on turbulent heat or mass transfer in a plain circular tube has been confined to problems with axisymmetric boundary conditions. But most practical cases have circumferential variations in heat or mass flux and, consequently, tangential variations in temperature or concentration. Such cases have been analysed by Reynolds [1] and Sparrow and Lin [2], whilst Sparrow and Black [3] gave experimental results for fully developed heat transfer to air in a tube with a circumferentially varying wall heat flux. Both [1] and [2] assumed $\varepsilon_{h,\omega}$ equal to $\varepsilon_{h,r}$ for all radii. But Sparrow and Black suggested it would be necessary to take a ratio of about ten near the wall in the analysis of [2] if it were to agree with their experiments. At the centre of the tube they suggested the ratio should be about unity. More recently, Quarmby and Anand [4] gave theory and experiment for non-symmetric mass transfer to air in a plain tube. They assumed a ratio of unity throughout. One question left unanswered by that work was whether the agreement between theory and experiment would have been as good had a ratio been used in the analysis which was not unity and perhaps something like that suggested by Sparrow and Black. Further calculations are given here, for one of the cases for which Quarmby and Anand gave experimental results, in which the effect of $\varepsilon_{d,\omega}/\varepsilon_{d,r}$ on the predicted concentration profile is discovered.

FORMULATION OF THE EQUATIONS

The development of the concentration profile resulting from a diametral line source in an impervious wall tube was analysed by [4]. It is most suitable for the present purpose since a solution may be obtained with a manageable number of terms and computation may be kept to a minimum. The dimensionless concentration, θ, is

$$\theta = 1 + \sum_{\lambda=1}^{\lambda=\infty} A_\lambda \cos(\lambda\omega) \sum_{n=1}^{n=\infty} Kn\, Rn \exp\left[-\frac{\alpha_n^2 x^+}{Re}\right] \qquad (1)$$

where Kn, Rn and α_n are eigen constants, eigen functions and eigen values of the Sturm–Liouville problem and A_λ Fourier coefficients. In solving for Kn, Rn and α_n it was necessary to assume some ratio between $\varepsilon_{d,\omega}$ and $\varepsilon_{d,r}$ and Quarmby and Anand chose $\varepsilon_{d,\omega} = \varepsilon_{d,r}$. But it seems that the ratio should be considered, more generally, as a function of radius, $F(r)$. The calculations presented here are identical to those of Quarmby and Anand except in the choice of $F(r)$.

RELATIONSHIP BETWEEN THE RADIAL AND TANGENTIAL EDDY-DIFFUSIVITIES

Turbulence measurements by Laufer [5] and others suggest that turbulence is anisotropic near the tube wall and the tangential velocity fluctuations are not damped out as much as the radial. Laufer's results suggest the tangential fluctuations are twice as great as the radial near the wall and, correspondingly, we might consider $F(r)$ to be two there also. Sparrow and Black suggest it might be ten there but both results indicate a value of unity at the centre. The form which $F(r)$ takes between these values may be such that it is greater than unity only in the sublayer and is unity across the main stream. On the other hand it may vary smoothly from its wall value to unity at the centre. There is thus some justification for choosing, from an infinity of possibilities, variations of $F(r)$ which are in line with these considerations. Hence, in addition to the assumption of [4], namely,

(a) $F(r)$ is unity,

the following assumptions seemed worth trying:

(b) $F(r)$ is ten at the wall and varies parabolically to unity at the centre where it has zero slope.

(c) The wall value is two but otherwise the variation is as in (b).

(d) The tangential eddy diffusivity is constant across the sublayer and equal to $\varepsilon_{d,r}$ at the edge of the sublayer. $F(r)$ is correspondingly unity up to and including the edge of the sublayer and then rises to become infinity at the wall.

This last assumption includes in it, as less significant cases, all assumptions in which $F(r)$ goes from unity at the edge of the sublayer to some finite value, such as ten or two, at the wall. In physical terms, it means that because $\varepsilon_{d,r}$ is zero at the wall $\varepsilon_{d,\omega}$ is not necessarily also zero there. In some preliminary calculations, finite wall values for $F(r)$ were tried but they were abandoned in favour of (d).

Figure 1 shows the four assumptions for the ratio of tangential to radial eddy diffusivity. They are shown against r^+ since this best allows the variation in the sublayer to be seen.

CALCULATIONS AND DISCUSSION

In the calculations of [4] it was found that the series solution for θ was sufficiently accurate, unless x^+ was very small, when only seven eigen functions were used for each value of λ. With the extension of the solution to several assumptions about $F(r)$ even this small number involves extensive calculation. It was easily established by simple trial and error that for $x^+ > 1$ and except very close to the wall the solution is accurate to 1 per cent or so if we take only the first and second eigen functions for $\lambda = 2$ and 4 and the first eigen function only for $\lambda = 6$, 8 and 10. The concentration profile may, thus, be calculated with sufficient accuracy for our purpose by taking only eight terms in equation (1).

Table 1 shows the effect of assumptions (b), (c) and (d) on the first eigen value and constant for $\lambda = 2$ for one of the cases, $Sc = 0{\cdot}7$ with $Re = 21\,000$, for which [4] gave theory and experiment. The values for (a) calculated by [4] are included for comparison. Calculations were made of the effect of each assumption on some of the higher order eigen values for other values of λ. In no case was the relative effect greater than that noted for $n = 1$ and $\lambda = 2$. Calculations were also made for $Re = 120\,000$ for each assumption. The results were not significantly different in kind from those for

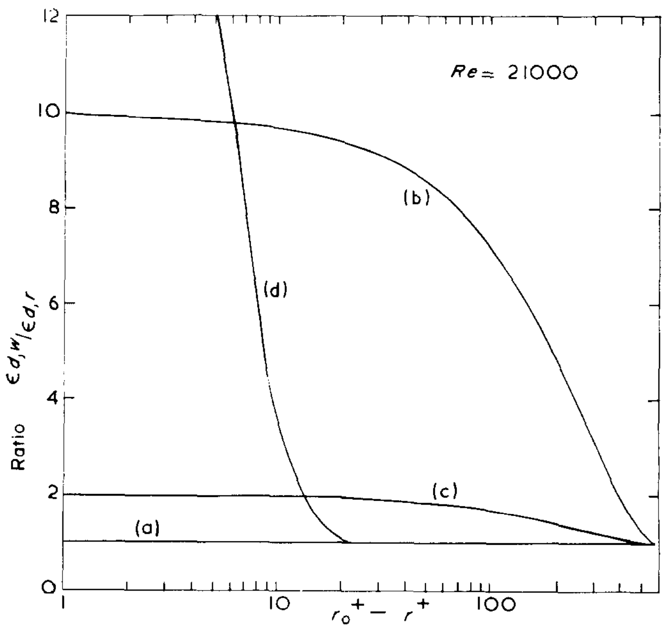

FIG. 1. Assumed ratios of $\varepsilon_{d,\omega}$ to $\varepsilon_{d,r}$.

$Re = 21\,000$. The results may be plotted to show the effect as a function of x^+, Fig. (2). The ordinate δ is the difference between the eigen function for the particular assumption and that for (a) divided by the latter. Since the first eigen function for $\lambda = 2$ makes the biggest contribution to the solution, these results allow us to decide which assumptions would have materially altered the agreement between theory and experiment achieved by [4].

It is often considered that the connection between the eddy diffusivities of heat or mass and that of momentum is a function of Prandtl or Schmidt Number. Clearly the Schmidt Number effect on the assumption for $F(r)$ should be considered as well as the Reynolds Number effect. Accordingly, calculations were made of the first eigen values and constants for $\lambda = 2$ for $Re = 21\,000$ but with $Sc = 0{\cdot}01$ and 1000. These are given in Table 1 and their effect as a function of x^+ plotted in Fig. 2. It is clear that for a Schmidt Number of 1000 there is a measurable effect resulting from

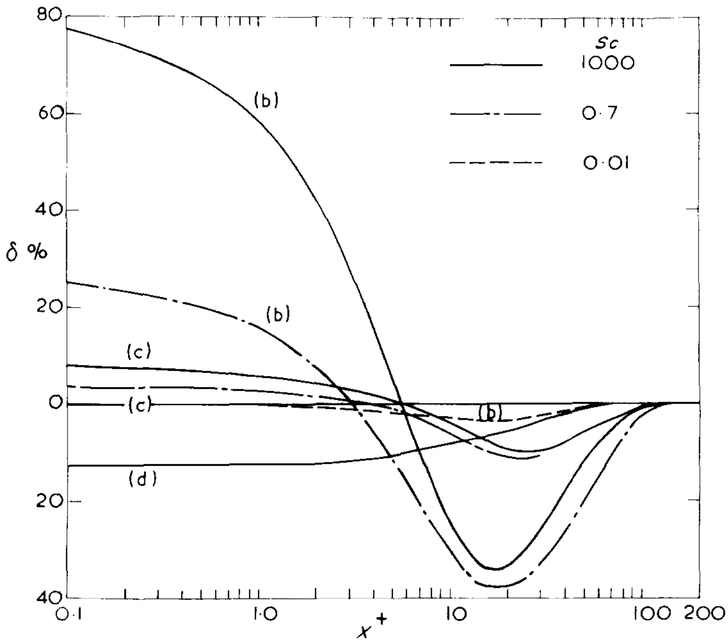

FIG. 2. Effect of assumptions for $\varepsilon_{d,\omega}$ and Schmidt Number on the first eigen function.

The effect of assumption (d) is negligible and does not show on the figure. That of (c) is about 5 per cent and that of (b) is about 20 per cent. The agreement between theory and experiment in [4] is much better than the latter and, perhaps, a little better than the former. Thus although the results of [4] may not establish that the ratio is exactly unity, it seems that it is not greater than two in the main stream. Sparrow and Black [3] concluded that their observations add strong support to a model of the transport process wherein the ratio is substantially greater than unity near the wall but is essentially unity at all other points in the flow. The present results are thus not inconsistent with Sparrow and Black because, as the results for assumption (d) show, the ratio may be infinite, even, in the sublayer without much affecting the predicted profile. The results do not lead to more definite ideas about the ratio in the sublayer.

Table 1. Effect of assumptions and Schmidt Number on the first eigen function and eigen constant

Schmidt Number		(a)	(b)	(c)	(d)
0·01	α_1	32·3164	33·5245	32·4533	32·3082
	K_1	1·21773	1·22597	1·21863	1·21797
0·7	α_1	18·2698	33·5915	20·7901	18·2994
	K_1	1·39519	1·75858	1·44774	1·39007
1000	α_1	21·0800	39·6022	24·1604	21·1696
	K_1	1·58623	2·84047	1·71108	1·37894

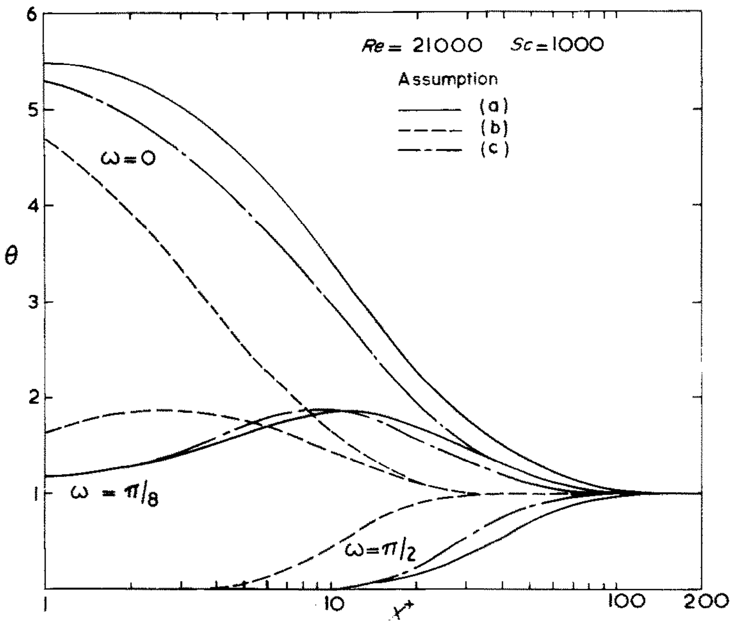

FIG. 3. Effect of assumptions on the axial development of the concentration profile; $Sc = 1000$, $Re = 21000$.

each of the assumptions. There is little or no effect for $Sc = 0{\cdot}01$. This is not surprising since in the governing differential equation the effective term is the sum of $\varepsilon_{d,\omega}/\nu$ and $1/Sc$. Clearly, for low Schmidt Number this sum will be much less affected by assumptions about $\varepsilon_{d,\omega}$.

The rest of the terms of the series for θ were calculated for $Sc = 1000$ and the effect of the assumptions (a), (b) and (c) on θ is shown in Fig. 3. It is clear that a suitable experiment with high Schmidt Number and low Reynolds Number could determine the truth or otherwise of the assumptions.

CONCLUSION

The development of a non-symmetric concentration profile, such as might result from a diametral line source in an impervious tube was calculated and the effect discovered of the assumption used about the ratio between the tangential and radial eddy diffusivities.

The ratios considered fully were

(i) the ratio is one throughout,
(ii) the ratio has a value two at the wall and one at the centre,
(iii) the ratio is ten at the wall and one at the centre,
(iv) the tangential eddy diffusivity is constant across the sublayer but the ratio is one elsewhere.

The effect of these ratios is very much dependent on the Schmidt Number.

The results for $Sc = 0{\cdot}01$ are little affected by the different assumptions. For each Schmidt Number the results for the last assumption, which is equivalent to taking a ratio of infinity at the wall are little different from the results for the ratio unity. With Schmidt Number 0·7 the predictions using the (i) and (ii) are close together but (iii) gives a measurably different result. Since a calculation using (i) gave good agreement with previous experiment it seems that (iii) may not be correct. The true variation is somewhere between (i) and (ii) in the main stream. The results of (iv) prevent reaching any conclusion about the ratio in the sublayer from previous experiment.

The effect of the different assumptions is greatest in the present results for Schmidt Number of 1000. Assumptions (i) and (ii) are quite distinguishable for this case. The true variation of the ratio of tangential to radial eddy diffusivity would probably be best determined by a comparison of theory and experiment for fluids with high Prandtl or Schmidt Numbers. However some information better than that available at present may be obtainable from fluids of moderate Prandtl Number if the Reynolds Number is low and the sublayer forms a measurable portion of the flow.

REFERENCES

1. W. C. REYNOLDS, Turbulent heat transfer in a circular tube with variable circumferential heat flux, *Int. J. Heat Mass Transfer* 6, 445 (1963).

2. E. M. Sparrow and S. H. Lin, Turbulent heat transfer in a tube with circumferentially varying temperature and heat flux, *Int. J. Heat Mass Transfer* **6**, 866 (1963).
3. E. M. Sparrow and A. W. Black, Experiments on turbulent heat transfer with circumferentially varying boundary conditions *Trans. Am. Soc. Mech. Engrs* **89C**, 258 (1967).
4. A. Quarmby and R. K. Anand, Non-axisymmetric turbulent mass transfer in a circular tube, *J. Fluid Mech.* **38**, 457 (1969).
5. J. Laufer, The structure of turbulence in fully developed pipe flow, NACA Report No. 1174 (1954).

Int. J. Heat Mass Transfer. Vol. 15, pp. 870–872. Pergamon Press 1972. Printed in Great Britain

ASYMPTOTIC SUCTION PROFILES IN FREE CONVECTION LAMINAR BOUNDARY LAYER FLOWS

D. E. BOURNE*

University of Sheffield, Sheffield, England

(*Received* 30 *September* 1971 *and in revised form* 15 *November* 1971)

INTRODUCTION

There are exact asymptotic solutions to several problems of laminar boundary layer flow over porous surfaces with uniform suction. The earliest, and one of the simplest recorded, is that of Griffith and Meredith [1] for two-dimensional incompressible flow over a semi-infinite flat plate. The asymptotic solution to the corresponding problem of compressible flow without heat transfer was obtained by Young [2], and this has been extended to flow with heat transfer by Lew and Fannucci [3].

Axially symmetric flows with uniform suction have also received some attention. Wuest [4], Lew [5] and Yasuhara [6] showed (independently) that for an incompressible flow along a circular cylinder, the axial component of velocity, w, has the simple asymptotic form

$$w = W[1 - (a/r)^R], \tag{1}$$

where a is the cylinder radius, r denotes distance from the axis, W is the axial component of velocity at large distances and

$$R = Va/\nu \tag{2}$$

is the Reynolds number associated with the suction velocity V, the radius of the cylinder and the kinematic viscosity ν. In a further contribution, Lew [7] extended the solution to compressible flow with heat transfer.

Flows involving free convection and suction appear to have received little attention. Eichhorn [8] obtained similarity solutions for free convective flow along a vertical flat plate with suitable non-uniform suction velocity and surface temperature. The only other contribution of this kind known to the present author is that of Kaloni [9] who obtained the asymptotic solution to a problem of flow of a visco-elastic fluid over a flat plate.

In each problem mentioned above, the asymptotic state is reached when vorticity generated at the boundary and diffusing away from it is exactly balanced by convection of vorticity towards the boundary brought about by the suction. For the steady asymptotic laminar state to be possible, the only condition on the suction velocity is that it should be positive.

This note is concerned largely with the problem of suction through the axially symmetric laminar boundary layer on a heated vertical circular cylinder, the axial component of flow being generated by natural convection. An interesting fact which emerges from the solution is that for a steady asymptotic laminar state to be possible it is not sufficient for the suction velocity, V, to be positive: it is also necessary to have $V > 2\kappa/a$, where κ is the thermal diffusivity and a is the cylinder radius.

ANALYSIS

Consider a semi-infinite circular cylinder, of radius a, placed with its axis vertical and its leading edge lowermost in a fluid of infinite extent. Let the cylinder be maintained at a uniform constant temperature T_1 and denote the ambient fluid temperature by T_0 ($< T_1$).

In addition to the natural convective flow which develops, suppose that there is a forced flow towards the cylinder due to extraction of fluid uniformly over the surface with velocity

* Senior Lecturer, Department of Applied Mathematics and Computing Science.